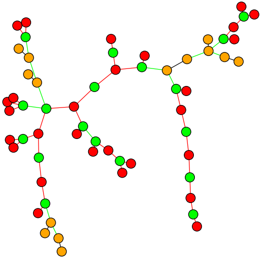
Cluster varieties for tree-shaped quivers and their cohomology



Frédéric Chapoton

CNRS & Université de Strasbourg

Octobre 2016

Cluster algebras and the associated varieties

Cluster algebras are commutative algebras
$\Longrightarrow$ cluster varieties (their spectrum) are algebraic varieties

Question: can we compute their cohomology rings ?

Cluster algebras and the associated varieties

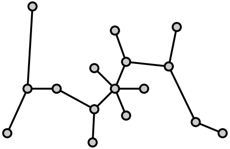
Cluster algebras are commutative algebras
$\implies$ cluster varieties (their spectrum) are algebraic varieties

Question: can we compute their cohomology rings ?

Why is this interesting ?
$\rightarrow$ classical way to study algebraic varieties
$\rightarrow$ useful (necessary) to understand integration on them
$\rightarrow$ answer is not obvious, and sometimes nice
$\rightarrow$ there are interesting known differential forms

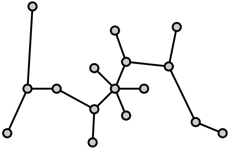
My choice goes to trees

Choice: try to handle first some simple cases
$\Longrightarrow$ restriction to quivers that are **trees**
(general quivers are more complicated)



My choice goes to trees

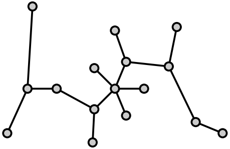
Choice: try to handle first some simple cases
$\Longrightarrow$ restriction to quivers that are **trees**
(general quivers are more complicated)



This choice is restrictive and rather arbitrary,
but turns out to involve a **nice combinatorics** of perfect matchings and independent sets in trees

My choice goes to trees

Choice: try to handle first some simple cases
$\Longrightarrow$ restriction to quivers that are **trees**
(general quivers are more complicated)



This choice is restrictive and rather arbitrary,
but turns out to involve a **nice combinatorics** of perfect matchings and independent sets in trees

$\rightarrow$ computing number of **points over finite fields** $\mathbb{F}_q$
can be seen as a first approximation towards determination of cohomology and is usually much more easy

First example (for babies)

Cluster algebra of type $\mathbb{A}_1$: $\boxed{\alpha} - x$ with one frozen vertex α.
Presentation by the unique relation

$$x\,x' = 1 + \alpha$$

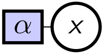
First example (for babies)

Cluster algebra of type $\mathbb{A}_1$: $\boxed{\alpha} - x$ with one frozen vertex α.
Presentation by the unique relation

$$x\,x' = 1 + \alpha$$

We will consider cluster algebras **with invertible coefficients**
So here α is assumed to be invertible.

First example (for babies)

Cluster algebra of type $\mathbb{A}_1$: 

with one frozen vertex α.
Presentation by the unique relation

$$x\,x' = 1 + \alpha$$

We will consider cluster algebras **with invertible coefficients**
So here α is assumed to be invertible.
One can then do **two different things**:
$\rightarrow$ (1) either let α be a coordinate, solve for α, and get the open sub-variety $xx' \neq 1$ with coordinates x, x'.
$\rightarrow$ (2) or fix α to a generic invertible value (here $\alpha \neq -1, 0$) and get the variety $x \neq 0$ with coordinate x.

First example (for babies)

$$x\,x' = 1 + \alpha \quad (*)$$

The first case (1) (α as variable) is a **cluster variety** spectrum of the cluster algebra $R = \mathbb{C}[x, x', \alpha, \alpha^{-1}]/(*)$

First example (for babies)

$$x\,x' = 1 + \alpha \quad (*)$$

The first case (1) (α as variable) is a **cluster variety**
spectrum of the cluster algebra $R = \mathbb{C}[x, x', \alpha, \alpha^{-1}]/(*)$

The second case (2) (α fixed to a generic value) could be called a **cluster fiber variety**:
the inclusion of algebras $\mathbb{C}[\alpha, \alpha^{-1}] \to R$
gives a projection of varieties $\mathbb{C}^* \leftarrow \mathit{Spec}(R)$
and one looks at the (generic) fibers of this **coefficient morphism**.

First example (for babies)

$$x\,x' = 1 + \alpha \quad (*)$$

The first case (1) (α as variable) is a **cluster variety**
spectrum of the cluster algebra $R = \mathbb{C}[x, x', \alpha, \alpha^{-1}]/(*)$

The second case (2) (α fixed to a generic value) could be called a **cluster fiber variety**:
the inclusion of algebras $\mathbb{C}[\alpha, \alpha^{-1}] \to R$
gives a projection of varieties $\mathbb{C}^* \leftarrow Spec(R)$
and one looks at the (generic) fibers of this **coefficient morphism**.

Note that the fiber at $\alpha = -1$ is singular.

General case of trees

Let us generalize this simple example.

For any tree, there is a well-defined cluster type
(because all orientations of a tree are equivalent by mutation)
one can therefore work with the alternating orientation, where every vertex is either a source or a sink

General case of trees

Let us generalize this simple example.

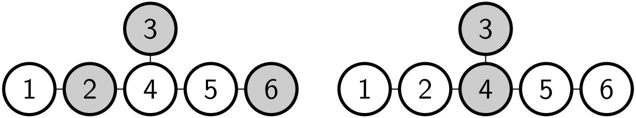
For any tree, there is a well-defined cluster type
(because all orientations of a tree are equivalent by mutation)
one can therefore work with the alternating orientation, where every vertex is either a source or a sink

For any tree T, the aim is to define several varieties
that are a kind of **compound** between cluster varieties and fibers

For that, need first to introduce some combinatorics on trees

Independent sets in graphs

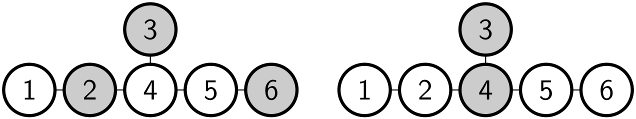
By definition, an **independent set** in a graph G is
a subset S of the set of vertices of G
such that every edge contains at most one element of S



independent and not independent

Independent sets in graphs

By definition, an **independent set** in a graph G is
a subset S of the set of vertices of G
such that every edge contains at most one element of S



independent and not independent

A **maximum independent set** is an independent set of maximal cardinality among all independent sets.
(not the same as being maximal for inclusion)

Independent sets in graphs

Independent sets are a very classical notion in graph theory.
$\rightarrow$ NP-complete problem for general graphs (Richard Karp, 1972)
$\rightarrow$ polynomial algorithm for bipartite graphs (Jack Edmonds, 1961).
$\rightarrow$ a very nice description for trees (Jennifer Zito 1991 ; Michel Bauer and Stéphane Coulomb 2004)

Independent sets in graphs

Independent sets are a very classical notion in graph theory.
$\rightarrow$ NP-complete problem for general graphs (Richard Karp, 1972)
$\rightarrow$ polynomial algorithm for bipartite graphs (Jack Edmonds, 1961).
$\rightarrow$ a very nice description for trees (Jennifer Zito 1991 ; Michel Bauer and Stéphane Coulomb 2004)

One has to distinguish three kinds of vertices:
- vertices belonging **to all** maximal independent sets: RED ●
- vertices belonging **to some (not all)** max. independent sets: ORANGE ●
- vertices belonging **to no** maximal independent set: GREEN ●

Chosen colors are "traffic light colors"

Independent sets in graphs

Independent sets are a very classical notion in graph theory.
$\rightarrow$ NP-complete problem for general graphs (Richard Karp, 1972)
$\rightarrow$ polynomial algorithm for bipartite graphs (Jack Edmonds, 1961).
$\rightarrow$ a very nice description for trees (Jennifer Zito 1991 ; Michel Bauer and Stéphane Coulomb 2004)

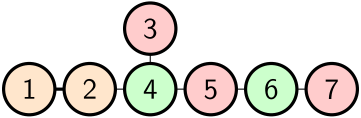
One has to distinguish three kinds of vertices:
- vertices belonging **to all** maximal independent sets: RED ●
- vertices belonging **to some (not all)** max. independent sets: ORANGE ●
- vertices belonging **to no** maximal independent set: GREEN ●

Chosen colors are "traffic light colors"

Nota Bene: this green has nothing to do with *green sequences*

Canonical coloring

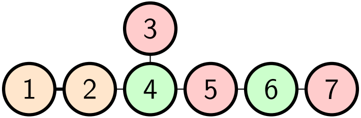
This gives a **canonical** coloring of every tree !
Here is one example of this coloring:



Here, there are 2 maximal indep. sets $\{1, 3, 5, 7\}$ and $\{2, 3, 5, 7\}$.

Canonical coloring

This gives a **canonical** coloring of every tree !
Here is one example of this coloring:

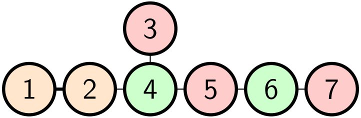


Here, there are 2 maximal indep. sets $\{1, 3, 5, 7\}$ and $\{2, 3, 5, 7\}$.

This coloring can be described by local "Feynman" rules:
- a green vertex has at least two red neighbors
- a red vertex has only green neighbors
- orange vertices are grouped into neighbourly pairs

Canonical coloring

This gives a **canonical** coloring of every tree !
Here is one example of this coloring:



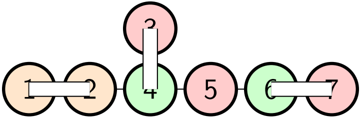
Here, there are 2 maximal indep. sets $\{1, 3, 5, 7\}$ and $\{2, 3, 5, 7\}$.

This coloring can be described by local "Feynman" rules:
- a green vertex has at least two red neighbors
- a red vertex has only green neighbors
- orange vertices are grouped into neighbourly pairs

It turns out that this coloring is also related to **matchings**.

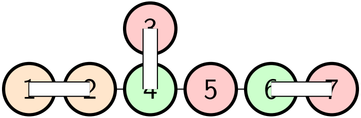
Coloring and matchings

A matching is a set of edges with no common vertices.
A **maximum matching** is a matching of maximum cardinality among all matchings.



Coloring and matchings

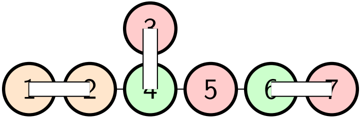
A matching is a set of edges with no common vertices.
A **maximum matching** is a matching of maximum cardinality among all matchings.



Other names: **dimer coverings** or **domino tilings**.
Here not required to cover all vertices (perfect matchings)

Coloring and matchings

A matching is a set of edges with no common vertices.
A **maximum matching** is a matching of maximum cardinality among all matchings.



Other names: **dimer coverings** or **domino tilings**.
Here not required to cover all vertices (perfect matchings)

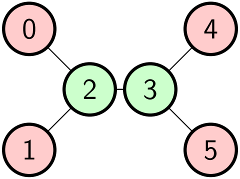
Theorem (Zito ; Bauer-Coulomb)

This coloring is the same as:

- *orange: vertices always in the same domino in all max. matchings*
- *green: vertices always covered by a domino in any max. matching*
- *red: vertices not covered by a domino in some max. matching*

Red-green components

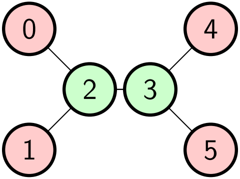
One can then use this coloring to define **red-green components**: keep only the edges linking a red vertex to a green vertex; this defines a forest; take its connected components



An example with two red-green components $\{0, 1, 2\}$ and $\{3, 4, 5\}$

Red-green components

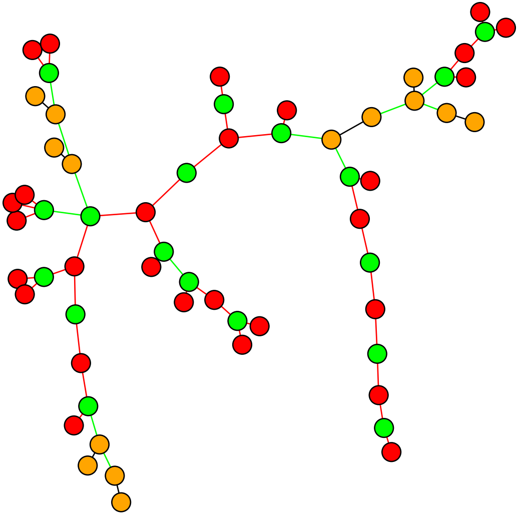
One can then use this coloring to define **red-green components**: keep only the edges linking a red vertex to a green vertex; this defines a forest; take its connected components



An example with two red-green components $\{0, 1, 2\}$ and $\{3, 4, 5\}$

For a tree T, let us call **dimension** $\dim T = \#$ red● $- \#$ green●.
This is always an integer $\dim(T) \geq 0$.
In the example above, the dimension is $4 - 2 = 2$.

Here is a big random example of tree, with its canonical coloring



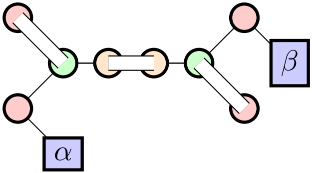
So what are the varieties ?

Take a tree T and consider the alternating (bipartite) orientation on T. This gives a quiver, initial data for cluster theory. Another orientation would give a quiver equivalent by mutation.

So what are the varieties ?

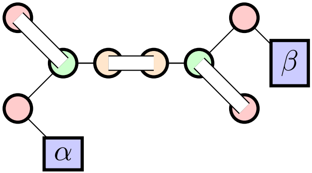
Take a tree T and consider the alternating (bipartite) orientation on T. This gives a quiver, initial data for cluster theory. Another orientation would give a quiver equivalent by mutation.

Pick a maximum matching of T and attach one frozen vertex to every vertex not covered by the matching.



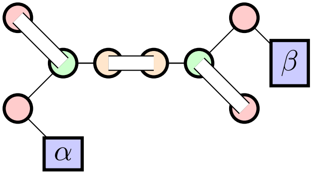
(Claim: no loss in generality compared to arbitrary coefficients)
$\rightarrow$ every coefficient is attached to a red vertex

So what are the varieties ?



$\rightarrow$ the extended graph is still a tree, and has a perfect matching.
$\rightarrow$ the number of frozen vertices is $\dim(T)$.

So what are the varieties ?



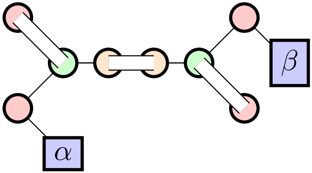
$\rightarrow$ the extended graph is still a tree, and has a perfect matching.
$\rightarrow$ the number of frozen vertices is $\dim(T)$.
Then **choose independently** for every red-green component:
- either to let all coefficients vary (but staying invertible)
- or to let all coefficients be fixed at generic (invertible) values

So what are the varieties ?

The equations are the cluster exchange relations for the alternating orientation (of the extended tree): $x_i x_i' = 1 + \prod_j x_j$.
One uses here a theorem of Berenstein-Fomin-Zelevinsky (in "Cluster Algebras III") which gives a presentation by generators and relations of **acyclic** cluster algebras.

So what are the varieties ?

The equations are the cluster exchange relations for the alternating orientation (of the extended tree): $x_i x_i' = 1 + \prod_j x_j$.
One uses here a theorem of Berenstein-Fomin-Zelevinsky (in "Cluster Algebras III") which gives a presentation by generators and relations of **acyclic** cluster algebras.



In this example, one can choose to fix β and let α vary.
This is really a mixture between the global cluster variety and the fibers of the coefficient morphism.

The matching does not matter

Theorem

This variety does not depend on the matching (up to isomorphism). All these varieties are smooth.

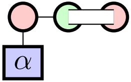
Proved using monomial isomorphisms ; smoothness by induction

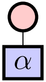
The matching does not matter

Theorem

This variety does not depend on the matching (up to isomorphism). All these varieties are smooth.

Proved using monomial isomorphisms ; smoothness by induction
Note that the genericity condition can be made very explicit and is really necessary to ensure smoothness: Counter examples





$\mathbb{A}_3$ singular when $\alpha = 1$ and $\mathbb{A}_1$ when $\alpha = -1$
($\mathbb{A}_1$ was the baby's example)

Points over finite fields

equations have coefficients in $\mathbb{Z} \to$ reduction to finite field $\mathbb{F}_q$.

Theorem

For X any of these varieties, there exists a polynomial P_X such that $\#X(\mathbb{F}_q)$ is given by $P_x(q)$.

This is a first hint that maybe cohomology may be "nice".
(not true for all algebraic varieties, see elliptic curves)

Points over finite fields

equations have coefficients in $\mathbb{Z} \to$ reduction to finite field $\mathbb{F}_q$.

Theorem

For X any of these varieties, there exists a polynomial P_X such that $\#X(\mathbb{F}_q)$ is given by $P_x(q)$.

This is a first hint that maybe cohomology may be "nice".
(not true for all algebraic varieties, see elliptic curves)
The proof is by induction on the size of trees. One picks a vertex, corresponding to a variable x. Then either $x = 0$ or $x \neq 0$. Both reduces to similar varieties (for well chosen vertices).

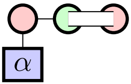
Points over finite fields

equations have coefficients in $\mathbb{Z} \to$ reduction to finite field $\mathbb{F}_q$.

Theorem

For X any of these varieties, there exists a polynomial P_X such that $\#X(\mathbb{F}_q)$ is given by $P_x(q)$.

This is a first hint that maybe cohomology may be “nice”.
(not true for all algebraic varieties, see elliptic curves)
The proof is by induction on the size of trees. One picks a vertex, corresponding to a variable x. Then either $x = 0$ or $x \neq 0$. Both reduces to similar varieties (for well chosen vertices).



For $\mathbb{A}_3$ with α generic, one gets $q^3 - 1$ points.

Free action of a torus

Recall the dimension $\dim(T) = \#\text{ red} - \#\text{ green}$
Clearly the dimension is additive over red-green components.

Free action of a torus

Recall the dimension $\dim(T) = \#\text{ red} - \#\text{ green}$
Clearly the dimension is additive over red-green components.

Consider the variety X_T associated with tree T and a choice for every red-green component of T between "varying" or "generic" coefficients. Let N be the sum of $\dim C$ over all "generic"-type red-green components C.

Theorem

There is a free action of $(\mathbb{C}^*)^N$ on X_T.
Moreover the enumerating polynomial P_X can be written as $(q-1)^N$ times a reciprocal polynomial.

Free action of a torus

Recall the dimension $\dim(T) = \#$ red $- \#$ green
Clearly the dimension is additive over red-green components.

Consider the variety X_T associated with tree T and a choice for every red-green component of T between "varying" or "generic" coefficients. Let N be the sum of dim C over all "generic"-type red-green components C.

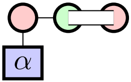
Theorem

There is a free action of $(\mathbb{C}^*)^N$ on X_T.
Moreover the enumerating polynomial P_X can be written as $(q-1)^N$ times a reciprocal polynomial.

Reciprocal means $P(1/q) = q^{-d} P(q)$ (palindromic coefficients)

Free action: an example

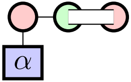
Let us look at the example of type $\mathbb{A}_3$ (with $\dim(T) = 1$):



$$xx' = 1 + \alpha y,$$
$$yy' = 1 + xz,$$
$$zz' = 1 + y.$$

Free action: an example

Let us look at the example of type $\mathbb{A}_3$ (with $\dim(T) = 1$):



$$xx' = 1 + \alpha y,$$
$$yy' = 1 + xz,$$
$$zz' = 1 + y.$$

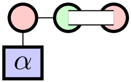
Here $N = 1$ and one can pick $\alpha = -1$ as generic value

Free action of $\mathbb{C}^*$ with coordinate λ:

$$x \mapsto \lambda x,$$
$$y \mapsto y,$$
$$z \mapsto z/\lambda.$$

Free action: an example

Let us look at the example of type $\mathbb{A}_3$ (with $\dim(T) = 1$):



$$xx' = 1 + \alpha y,$$
$$yy' = 1 + xz,$$
$$zz' = 1 + y.$$

Here $N = 1$ and one can pick $\alpha = -1$ as generic value
Free action of $\mathbb{C}^*$ with coordinate λ:

$$x \mapsto \lambda x,$$
$$y \mapsto y,$$
$$z \mapsto z/\lambda.$$

The enumerating polynomial is $q^3 - 1 = (q-1)(q^2 - q + 1)$
This variety is not a product, but a non-trivial $\mathbb{C}^*$-principal bundle.

What about cohomology ?

Tools that can be used to study cohomology :
- algebraic de Rham cohomology (algebraic differential forms)
- cohomology with compact support
- mixed Hodge structure, weights on cohomology

What about cohomology ?

Tools that can be used to study cohomology :
- algebraic de Rham cohomology (algebraic differential forms)
- cohomology with compact support
- mixed Hodge structure, weights on cohomology

Lemma

For two adjacent vertices x, y, either $x \neq 0$ or $y \neq 0$.

Just because of the exchange relation $xx' = 1 + *y$!
$\Longrightarrow$ coverings by two open sets, can apply Mayer-Vietoris long exact sequence

What about cohomology ?

Tools that can be used to study cohomology :
- algebraic de Rham cohomology (algebraic differential forms)
- cohomology with compact support
- mixed Hodge structure, weights on cohomology

Lemma

For two adjacent vertices x, y, either $x \neq 0$ or $y \neq 0$.

Just because of the exchange relation $xx' = 1 + *y$!
$\Longrightarrow$ coverings by two open sets, can apply Mayer-Vietoris long exact sequence
can also find coverings by more open sets $\rightarrow$ use spectral sequences.
The Hodge structure sometimes help to prove that the spectral sequence degenerates at step 2.

Some simple classes in cohomology

With all these tools , only partial results.
For every "varying" coefficient α, there is a class $\frac{d\alpha}{\alpha}$.

Some simple classes in cohomology

With all these tools , only partial results.
For every "varying" coefficient α, there is a class $\frac{d\alpha}{\alpha}$.

For every tree, there is a natural 2-form usually called the **Weil-Petersson** form (Gekhtman-Schapiro-Vainshtein, Fock-Goncharov, G. Muller)

$$\mathrm{WP} = \sum_{i \to j} \frac{dx_i dx_j}{x_i x_j}.$$

sum running over edges in the frozen quiver, excluding the fixed coefficients.

Some simple classes in cohomology

With all these tools , only partial results.
For every "varying" coefficient α, there is a class $\frac{d\alpha}{\alpha}$.

For every tree, there is a natural 2-form usually called the **Weil-Petersson** form (Gekhtman-Schapiro-Vainshtein, Fock-Goncharov, G. Muller)

$$\mathrm{WP} = \sum_{i \to j} \frac{dx_i dx_j}{x_i x_j}.$$

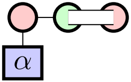
sum running over edges in the frozen quiver, excluding the fixed coefficients.

Not enough

The sub-algebra generated by those forms is **not** the full cohomology ring in general !

There are other classes in cohomology

Example of type $\mathbb{A}_3$

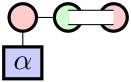


• For α invertible variable, one-form $\frac{d\alpha}{\alpha}$ and 2-form WP $= \frac{dxd\alpha}{x\alpha} + \frac{dxdy}{xy} + \frac{dzdy}{zy}$ do generate all the cohomology

$$H^* = \mathbb{Q}, \quad \mathbb{Q}, \quad \mathbb{Q}, \quad \mathbb{Q}, \quad \mathbb{Q}.$$

There are other classes in cohomology

Example of type $\mathbb{A}_3$



• For α invertible variable, one-form $\frac{d\alpha}{\alpha}$ and 2-form $\mathsf{WP} = \frac{dxd\alpha}{x\alpha} + \frac{dxdy}{xy} + \frac{dzdy}{zy}$ do generate all the cohomology

$$H^* = \mathbb{Q}, \quad \mathbb{Q}, \quad \mathbb{Q}, \quad \mathbb{Q}, \quad \mathbb{Q}.$$

• For α generic fixed, $\mathsf{WP} = \frac{dxdy}{xy} + \frac{dzdy}{zy}$, but cohomology has dimensions

$$H^* = \mathbb{Q}, \quad 0, \quad \mathbb{Q}, \quad \mathbb{Q}^2$$

Mixed Tate-Hodge structures

They form an Abelian category, with a forgetful functor to $\mathbb{Q}$-vector spaces, and with one simple object $\mathbb{Q}(i)$ for every $i \in \mathbb{Z}$
no morphisms $\mathbb{Q}(i) \to \mathbb{Q}(j)$ if $i \neq j$.
Some extensions $\mathbb{Q}(i) \to E \to \mathbb{Q}(j)$ if $j > i$.
Think of representations of a hereditary quiver with vertices $Q_0 \simeq \mathbb{Z}$

Mixed Tate-Hodge structures

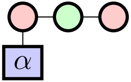
They form an Abelian category, with a forgetful functor to $\mathbb{Q}$-vector spaces, and with one simple object $\mathbb{Q}(i)$ for every $i \in \mathbb{Z}$ no morphisms $\mathbb{Q}(i) \to \mathbb{Q}(j)$ if $i \neq j$.
Some extensions $\mathbb{Q}(i) \to E \to \mathbb{Q}(j)$ if $j > i$.
Think of representations of a hereditary quiver with vertices $Q_0 \simeq \mathbb{Z}$
One can find such structure on the cohomology of all these varieties.
Deligne (and many famous names involved) gives us a mixed Hodge structure on the cohomology of all algebraic varieties.

Mixed Tate-Hodge structures

They form an Abelian category, with a forgetful functor to $\mathbb{Q}$-vector spaces, and with one simple object $\mathbb{Q}(i)$ for every $i \in \mathbb{Z}$ no morphisms $\mathbb{Q}(i) \to \mathbb{Q}(j)$ if $i \neq j$.
Some extensions $\mathbb{Q}(i) \to E \to \mathbb{Q}(j)$ if $j > i$.
Think of representations of a hereditary quiver with vertices $Q_0 \simeq \mathbb{Z}$
One can find such structure on the cohomology of all these varieties.
Deligne (and many famous names involved) gives us a mixed Hodge structure on the cohomology of all algebraic varieties.
One can prove by induction that it is Hodge-Tate in the varieties under consideration. This means that there are no "more complicated factors".

Mixed Tate-Hodge structures: one example

Consider the type $\mathbb{A}_3$ for generic α

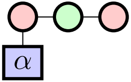


$$H^* = \mathbb{Q}(0), \quad 0, \quad \mathbb{Q}(2), \quad \mathbb{Q}(2) \oplus \mathbb{Q}(3)$$

the last summand H^3 is not pure (has several weights)

Mixed Tate-Hodge structures: one example

Consider the type $\mathbb{A}_3$ for generic α



$$H^* = \mathbb{Q}(0), \quad 0, \quad \mathbb{Q}(2), \quad \mathbb{Q}(2) \oplus \mathbb{Q}(3)$$

the last summand H^3 is not pure (has several weights)

Knowing this decomposition allows to recover the number of points over finite fields. Essentially every direct summand $\mathbb{Q}(i)$ in the cohomology group H^j gives a summand $(-1)^j q^i$.
(But beware that one must use cohomology with compact support).

The cohomological information above gives back $q^3 - 1$.

Some results (Dynkin diagrams are trees)

- For type $\mathbb{A}_n$ with n even, every class is a power of the Weil-Petersson 2-form.

Some results (Dynkin diagrams are trees)

- For type $\mathbb{A}_n$ with n even, every class is a power of the Weil-Petersson 2-form.
- For type $\mathbb{A}_n$ with n odd and one varying coefficient α, every class is in the ring generated by WP and $\frac{d\alpha}{\alpha}$.

Some results (Dynkin diagrams are trees)

- For type $\mathbb{A}_n$ with n even, every class is a power of the Weil-Petersson 2-form.
- For type $\mathbb{A}_n$ with n odd and one varying coefficient α, every class is in the ring generated by WP and $\frac{d\alpha}{\alpha}$.
- For type $\mathbb{A}_n$ with n odd and one generic coefficient α, only a guess: Besides powers of WP, there are $(n+1)/2$ more forms in top degree, with distinct weights.

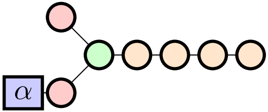
Some results (Dynkin diagrams are trees)

- For type $\mathbb{A}_n$ with n even, every class is a power of the Weil-Petersson 2-form.
- For type $\mathbb{A}_n$ with n odd and one varying coefficient α, every class is in the ring generated by WP and $\frac{d\alpha}{\alpha}$.
- For type $\mathbb{A}_n$ with n odd and one generic coefficient α, only a guess: Besides powers of WP, there are $(n+1)/2$ more forms in top degree, with distinct weights.
- For type $\mathbb{D}_n$ with n odd and one generic coefficient, one can show that besides powers of WP, there is one more form in degree 3 and one more in degree n.

Some results (Dynkin diagrams are trees)

- For type $\mathbb{A}_n$ with n even, every class is a power of the Weil-Petersson 2-form.
- For type $\mathbb{A}_n$ with n odd and one varying coefficient α, every class is in the ring generated by WP and $\frac{d\alpha}{\alpha}$.
- For type $\mathbb{A}_n$ with n odd and one generic coefficient α, only a guess: Besides powers of WP, there are $(n+1)/2$ more forms in top degree, with distinct weights.
- For type $\mathbb{D}_n$ with n odd and one generic coefficient, one can show that besides powers of WP, there is one more form in degree 3 and one more in degree n.
- Something can also be said about some trees of general shape H, in particular for $\mathbb{E}_6$ and $\mathbb{E}_8$
- case $\mathbb{E}_7$ with generic coefficient not fully understood.

Some details on type $\mathbb{D}$



One concrete example : $\mathbb{D}_n$ with n odd and generic coefficient α

Theorem

The cohomology is given by

$$\begin{cases} \mathbb{Q}(k) & \text{if } k = 0 \mod (2) \\ \mathbb{Q}(k-1) & \text{if } k = 1 \mod (2) \text{ and } k \neq 1, n \\ \mathbb{Q}(n-1) \oplus \mathbb{Q}(n) & \text{if } k = n \end{cases}$$

For $n = 3$, this coincide with the answer for $\mathbb{A}_3$, as it should.

Some things being skipped today

$\rightarrow$ Results on counting points over $\mathbb{F}_q$ (nice formulas)
Just one tiny example in type $\mathbb{E}_n$ for n even:

$$(q^2 - q + 1)\frac{(q^{n-1} - 1)}{(q - 1)}$$

$\rightarrow$ cellular decomposition (when coefficients are variables)
$\implies$ formula as a sum for the number of points over $\mathbb{F}_q$.

$\rightarrow$ Simple algorithm to compute the coloring.

The mysterious Poincaré duality

and the "curious Lefschetz property of Hausel and Rodriguez-Villegas"

Recall that the enumerating polynomial P_X can be written as $(q-1)^N$ times a reciprocal polynomial, because there is a free torus action.

The mysterious Poincaré duality

and the "curious Lefschetz property of Hausel and Rodriguez-Villegas"

Recall that the enumerating polynomial P_X can be written as $(q-1)^N$ times a reciprocal polynomial, because there is a free torus action.

It seems that the quotient varieties are very interesting. Their enumerating polynomials are palindromic. Why ?

The mysterious Poincaré duality

and the "curious Lefschetz property of Hausel and Rodriguez-Villegas"

Recall that the enumerating polynomial P_X can be written as $(q-1)^N$ times a reciprocal polynomial, because there is a free torus action.
It seems that the quotient varieties are very interesting. Their enumerating polynomials are palindromic. Why ?

Speyer and Lam (in a more general context) have proved that the palindromy comes from a structural property of the cohomology.

Namely, cup-product by the Weil-Petersson 2-form gives an isomorphism between some subspaces of the cohomology.

This is very similar to the classical statement of algebraic geometry, where the hyperplane class of a projective algebraic variety acts on the cohomology in a way that implies usual Poincaré duality.

Some perspectives (many things to do)

- at least complete the case of type $\mathbb{A}$ and Dynkin diagrams

- go beyond trees to all acyclic quivers and general matrices
(see article by David E Speyer and Thomas Lam, arxiv:1604.06843.)

- say something about the integrals ($\zeta(2)$ and $\zeta(3)$ are involved)

- try to organize all the cohomology rings of type $\mathbb{A}$ into some kind of algebraic structure (Hopf algebra, operad ?)

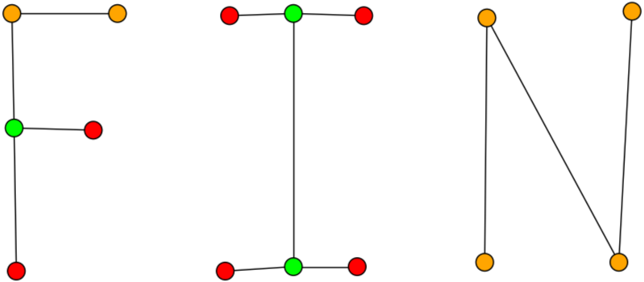
- study the topology of the real points (in relation with $q = -1$)

- topology of the set of non-generic parameters (toric arrangement)

- what about K-theory instead of cohomology ?

- understand the mysterious palindromic/Poincaré/Lefschetz property

- some amusing relations between P_X and Pisot numbers



Grazie